Coleman–de Luccia Instanton of the Second Order in a Brane World

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The second order Coleman–de Luccia instanton and its action in the Randall–Sundrum type II model are investigated and the comparison with the results in Einstein's general relativity is done in the present paper.

1. INTRODUCTION

The instability of the false vacuum of a scalar field interacting with gravity can result in the formation of rapidly expanding bubbles containing the scalar field (inflaton) on the true vacuum side of the potential barrier (Coleman and de Luccia, 1980). This process, described by the Coleman–de Luccia (CdL) instanton, takes place (the CdL instanton exists) under some conditions on the inflaton potential V (Jensen and Steinhardt, 1984; Tanaka, 1999; Balek and Demetrian, 2004). There is also another way in which the false vacuum can decay. This one is mediated by the Hawking–Moss (HM) instanton which exists for any non-negative potential V obeying two non-degenerate minima separated by a finite barrier. The false vacuum decay via the HM instanton describes the inflaton that jumps at the top of the potential barrier within the horizon-size domain (Hawking and Moss, 1982; Linde, 1990), and afterwards rolls down-hill to the true vacuum.

The semiclassical theory of the false vacuum decay by the instantons can be formulated also in the brane world scenarios, for further details see Bouhmadi-López *et al*. (2002), del Campo *et al*. (2004) and Demetrian (2005). The equations for the CdL instantons in the Randall–Sundrum type II scenario have the form

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(del Campo *et al.*, 2004)

$$a'' = -C \left\{ (\Phi')^2 + V + \frac{1}{8\sigma} \left[(5(\Phi')^2 + 2V)(-(\Phi')^2 + 2V) \right] \right\} a,$$

$$\Phi'' + 3\frac{a'}{a}\Phi' - \partial_\Phi V = 0, \tag{1}$$

where C is the constant equal to $8\pi/3$, $a = a(\tau)$ and $\Phi = \Phi(\tau)$ are the scale parameter and the inflaton, respectively, and σ stands for the brane tension. If the brane tension tends infinity (physically, if $V_{\text{char}}/\sigma \ll 1$, where V_{char} is a characteristic value of V along the solutions $\Phi(\tau)$) one obtains the standard equations for the CdL instantons in the Einstein's theory of relativity. The solution of the system (1) is called the CdL instanton if the (Euclidean) action

$$S = 2\pi^2 \int_0^{\tau_f} d\tau \left[a^3 \left(\frac{1}{2}(\Phi')^2 + V \right) \right.$$

$$\left. + \frac{a^3}{2\sigma} \left(\frac{1}{2}(\Phi')^2 + V \right)^2 + \frac{1}{C}(a^2 a'' + a(a')^2 - a) \right] \tag{2}$$

is finite. The finiteness of the action is guaranteed by the following boundary conditions

$$a(0) = 0, \quad a'(0) = 1, \quad \Phi'(0) = \Phi'(\tau_f) = 0, \tag{3}$$

where $\tau_f > 0$ is to be determined from $a(\tau_f) = 0$. The action (2) of the CdL instanton is considerably simplified by using the equations of motion (1) (Demetrian, 2005)

$$S = -\frac{4\pi^2}{3C} \int_0^{\tau_f} a\, d\tau - \frac{\pi^2}{\sigma} \int_0^{\tau_f} a^3(\Phi')^4 d\tau. \tag{4}$$

The existence and the properties of a solution to the system (1), (3) is a very interesting question (Jensen and Steinhardt, 1984; Tanaka, 1999; Hackworth and Weinberg, 2005; Demetrian, 2005), also in the limit $\sigma \to \infty$. For the definiteness, let us denote by Φ_M the value of the inflaton where V reaches the local maximum (the top) and $V_M \equiv V(\Phi_M)$ is the related energy density of the scalar field. There is always the trivial solution called HM instanton

$$\Phi = \Phi_M, \; a = \hat{H}_M^{-1} \sin(\hat{H}_M \tau), \quad \tau \in [0, \pi/\hat{H}_M]$$

$$\hat{H}_M^2 = \frac{8\pi}{3} \left(V_M + \frac{V_M^2}{2\sigma} \right) = H_M^2 \left(1 + \frac{V_M}{2\sigma} \right), \tag{5}$$

with the action (Demetrian, 2005)

$$S_{\text{HM}} = -\frac{\pi}{\hat{H}_M^2} = -\frac{3}{8V_M} \frac{1}{1 + \frac{V_M}{2\sigma}}, \tag{6}$$

where $H_M = (CV_M)^{1/2}$ is the Hubble parameter of the de Sitter universe filled by the energy density V_M and above defined $\hat{H}_M$ is the analogue of this quantity on considered brane. In the work of reference (Demetrian, 2005) it is shown that for

$$\hat{\xi} \equiv -\frac{1}{\hat{H}_M^2} \frac{d^2 V(\Phi)}{d\Phi^2}\bigg|_{\Phi=\Phi_M} \rightarrow l(l+3), \quad l = 1, 2, 3, \dots \tag{7}$$

there is the solution of the CdL instanton equations such that the function $\Phi(\tau)$ crosses the top of the barrier just l times—the lth order CdL instanton. (An analogical situation is also in the standard Einstein's theory of relativity, with only change $\hat{H}_M \mapsto H_M$ (Tanaka, 1999; Balek and Demetrian, 2004, 2005; Hackworth and Weinberg, 2005)). In the case $l = 1$ the situation has been studied extensively in Demetrian (2005), where the approximative formulas for the first order CdL instanton and its action have been found in the mentioned limit (7). Let us recall the main results briefly. If we introduce the notation $y = \Phi - \Phi_M$, then in the limit (7) we have the CdL instanton that is close to the HM instanton (the limit CdL instanton) and in the lowest order we can write $y = k \cos(\hat{H}_M \tau)$ and the constant k (the inflaton amplitude) can be determined perturbatively. The result is (Demetrian, 2005)

$$k^2 = -\frac{7(4 - \hat{\xi})}{16C + \frac{1}{24}\hat{\eta}^2 + \frac{1}{2}\hat{\zeta} + \frac{1}{32}\frac{V_M}{\sigma}(435 - 69C)} \equiv -\frac{7(4 - \hat{\xi})}{\mathfrak{D}}, \tag{8}$$

where we have introduced the following quantities describing the properties of the potential V at its top:

$$\hat{\eta} = \frac{1}{\hat{H}_M^2} \frac{d^3 V(\Phi)}{d\Phi^3}\bigg|_{\Phi=\Phi_M}, \qquad \hat{\zeta} = \frac{1}{\hat{H}_M^2} \frac{d^4 V(\Phi)}{d\Phi^4}\bigg|_{\Phi=\Phi_M}. \tag{9}$$

And the action of this instanton is given by the formula (Demetrian, 2005)

$$S_{CdL} = S_{HM} - \frac{4\pi^2}{3C} \frac{4C - 5}{3} \frac{1}{\hat{H}_M^2} \frac{V_M}{\sigma} k^2$$

$$= S_{HM} + \frac{4\pi^2}{3C} \frac{4C - 5}{3} \frac{1}{\hat{H}_M^2} \frac{V_M}{\sigma} \frac{7(4 - \hat{\xi})}{\mathfrak{D}}. \tag{10}$$

There are two possibilities: if $\mathfrak{D} > 0$ then the limit CdL instanton exists for $\hat{\xi} > 4$ and if $\mathfrak{D} < 0$ then the limit CdL instanton exists for $\hat{\xi} < 4$. Anyway, the action of such a limit CdL instanton is less than the action of related HM instanton and therefore the CdL instanton mediates the vacuum decay. The equations (8) and (10) can be compared with their counterpart in the general theory of relativity (Balek and Demetrian, 2004, 2005). The expression (8) for the inflatonamplitude

approaches the general-relativistic value (Balek and Demetrian, 2005) as $V_M/\sigma \to 0$.

2. THE LIMIT CDL INSTANTON OF THE SECOND ORDER

In order to investigate the limit CdL instanton of the second order we introduce new independently variable

$$x = \hat{H}_M \tau$$

and the expansions of the relevant quantities into the perturbative series in powers of the parameter k

$$y(x) = \sum_n k^n u_n(x) \qquad a(x) = \hat{H}_M^{-1} \sum_n k^n v_n(x)$$

$$\hat{\xi} = 10 + \sum_n k^n \Delta_n \qquad x_f = \pi + \sum_n k^n x_f^{(n)}. \tag{11}$$

Inserting these series into the equations of motion (1) we obtain the infinite system of linear equations for the functions u_n and v_n

$$u_n''(x) + 3\cot(x)u_n'(x) + 10u_n(x) = \mathcal{U}_n(x)$$

$$v_n''(x) + v_n(x) = \mathcal{V}_n(x)\sin(x),$$

where the source terms proportional to $\mathcal{U}_n$ and $\mathcal{V}_n$ are to be computed order-by-order using (11) and the Taylor expansion of the potential V around Φ_M

$$V = \hat{H}_M^2 \left[\frac{V_M}{\hat{H}_M^2} - \frac{1}{2}\hat{\xi}y^2 + \frac{1}{6}\hat{\eta}y^3 + \frac{1}{24}\hat{\zeta}y^4 + \cdots \right]$$

In the lowest order in k we obtain easily

$$v_0 = \sin(x) \qquad u_0 = 0 \qquad v_1 = 0. \tag{12}$$

The equation for u_1 is

$$u_1''(x) + 3\cot(x)u_1'(x) + 10u_1(x) = 0.$$

The solution must obey the boundary conditions (3). In order to interpret k as the amplitude of the limit instanton in the Φ-direction we can request that $|u_1(0)|(= |u_1(\pi)|) = 1$. Then we have

$$u_1(x) = \frac{1}{4}(5\cos^2(x) - 1). \tag{13}$$

One straightforwardly obtains also the equation for v_2

$$v_2'' + v_2 = -\left[E(u_1')^2 - Fu_1^2\right]\sin(x),$$

with

$$E = C + \frac{5 - C}{4}\frac{V_M}{\sigma}, \qquad F = 5C\left(1 + \frac{V_M}{\sigma}\right).$$

The boundary conditions (3) require that $v(0) = v'(0) = 0$, therefore

$$v_2(x) = \frac{1}{6144}\left[24(100E - 13F)x\cos(x) + (-2800E + 752F)\sin(x) + \right.$$

$$15(20E - 7F)\sin(3x) - 25(4E + F)\sin(5x)\left.\right]. \tag{14}$$

Having this result we can compute the shift of the right-end point x_f in the second order in k. The result reads

$$x_f^{(2)} = \frac{\pi}{64}\left[-25E + \frac{13}{4}F\right] = \frac{5\pi}{256}\left[-7C + (-25 + 18C)\frac{V_M}{\sigma}\right]. \tag{15}$$

The explicit formula for the inflaton amplitude can be obtained from the equation for u_2 because Δ_1 enters this equation nontrivially. In fact, this equation reads

$$u_2''(x) + 3\cot(x)u_2'(x) + 10u_2(x)$$

$$= -\frac{\Delta_1}{4}(5\cos^2(x) - 1) + \frac{\hat{\eta}}{32}(5\cos^2(x) - 1)^2, \tag{16}$$

or, with the help of more appropriate independently variable $z = \cos(x)$, (16) has the form of hypergeometric equation:

$$(1 - z^2)\frac{d^2u_2(z)}{dz^2} - 4z\frac{du_2(z)}{dz} + 10u_2(z) = -\frac{\Delta_1}{4}(5z^2 - 1) + \frac{\hat{\eta}}{32}(5z^2 - 1)^2.$$

This equation has only polynomial solution that is bounded at $z = \pm 1$, namely

$$u_2 = \alpha + \beta z^4$$

where the constants α, β and the shift of the effective curvature of the potential Δ_1 must obey equations

$$-18\beta = \frac{25}{32}\hat{\eta}$$

$$12\beta = -\frac{5}{4}\Delta_1 - \frac{5}{16}\hat{\eta}$$

$$10\alpha = \frac{1}{4}\Delta_1 + \frac{1}{32}\hat{\eta}.$$

One obtains easily the solution:

$$\alpha = \frac{7}{960}\hat{\eta} \qquad \beta = -\frac{25}{576}\hat{\eta} \qquad \Delta_1 = \frac{1}{6}\hat{\eta}.$$

Now, we can conclude that the the function u_2 is given by the formula

$$u_2(x) = \frac{\hat{\eta}}{192}\left(\frac{7}{5} - \frac{25}{3}\cos^4(x)\right) \tag{17}$$

and the inflaton amplitude is given by

$$\hat{\xi} = 10 + k\Delta_1 \qquad \Rightarrow \qquad k = \frac{6}{\hat{\eta}}(\hat{\xi} - 10). \tag{18}$$

The result (18) is not affected by the presence of the term(s) in the instanton equations (1) proportional to the fraction V_M/σ. To compare this with the situation in the Einstein's theory of relativity see Balek and Demetrian (2005), and Demetrian (2004). The formula (18) is singular for $\hat{\eta} = 0$. If this situation takes place, we must go back to the equation (16) from which we see that the requirement of the absence of resonance gives us $\Delta_1 = 0$. This means that k remains undetermined in this case and one should continue the computation to higher order in k.

We will finish this section with the computation of the difference between the actions of the above studied second order CdL instanton and the related HM instanton. Both actions are given by the formula (4). In the lowest non-zero order the mentioned difference of the actions is

$$\delta S \equiv S_{\text{lim}}^{(l=2)} - S_{\text{HM}} = -\frac{4\pi^2}{3C}k^2\int_0^{\tau_f}\hat{H}_M^{-1}v_2(\hat{H}_M\tau)d\tau$$

$$= -\frac{4\pi^2}{3C\hat{H}_M^2}k^2\int_0^\pi v_2(x)dx = -\frac{4\pi^2}{3C\hat{H}_M^2}k^2\frac{25}{12}(C-1)\frac{V_M}{\sigma}$$

$$= -\frac{100\pi^2}{C\hat{H}_M^2}(C-1)\left(\frac{\hat{\xi} - 10}{\hat{\eta}}\right)^2\frac{V_M}{\sigma}. \tag{19}$$

The proportionality of δS to V_M/σ was expected since in the Einstein's relativity theory the analogical quantity is equal to zero (Demetrian, 2004; Balek and Demetrian, 2005), in the second order in k. δS diverges as $\hat{\eta} \to 0$—this is the same situation as within Einstein's theory of relativity.

3. CONCLUSION

We have derived explicit formula for the second order CdL instanton in the situation when the effective curvature of the inflaton potential at its top is close to the critical value 10. We have shown that the shape of the instanton in our

brane-world model is only small deformation of the shape of corresponding CdL instanton in the Einstein theory of relativity. This deformation can be described by two changes in the parameters describing the function $V = V(\Phi)$ close its local maximum: $\xi \mapsto \hat{\xi}$ and $\eta \mapsto \hat{\eta}$, where the *hat*-quantities are affected by the dimensionless fraction V_M/σ (as introduced in (5,7,9)) that describes the effect of the brane tension. We were also able to derive an explicit expression for the action of the instanton and we have shown that in our brane-world model the difference between the action of our instanton and related HM instanton appear in lower order in the k-expansion with respect to the situation in the general theory of relativity.

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REFERENCES

Balek, V. and Demetrian, M. (2004). Criterion for bubble formation in a de Sitter universe. *Phys. Rev. D* **69**, 063518.

Balek, V. and Demetrian, M. (2005). Euclidean action for vacuum decay in a de Sitter universe. *Phys. Rev. D* **71**, 023512.

Bouhmadi-López, M., González-Díaz, P. F., and Zhuk, A. (2002). On new gravitational instantons describing creation of brane-worlds. *Class. Quant. Grav.* **19**, [4863, hep-th/0208226].

Bucher, M., Goldhaber, A. S., and Turok, N. (1995). Open universe from inflation. *Phys. Rev. D* **52**, 3314–3337.

Coleman, S. and de Luccia, F. (1980). Gravitational effects on and of vacuum decay. *Phys. Rev. D* **21**, 3305–3315.

del Campo, S., Herrera, R., and Saavedra, J. (2004). Open inflationary universes in a brane world cosmology. *Phys. Rev. D* **70**, 023507, [hep-th/0404148].

Demetrian, M. (2004). Study of the Coleman–de Luccia instanton of the second order, [gr-qc/0408012].

Demetrian, M. (2005). False vacuum decay in a brane world cosmological model, to appear in *Gen. Rel. Grav.*, [gr-qc/0506028].

Guth, A. H. (1981). Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **23**, 347–356.

Hackworth, J. C. and Weinberg, E. J. (2005). Oscillating bounce solutions and vacuum tunneling in de Sitter spacetime. *Phys Rev. D* **71**, 044014.

Hawking, S. W. and Moss, I. G. (1982). Supercooled phase transitions in the very early universe. *Phys. Lett.* **101B**, 35–37.

Jensen, L. G. and Steinhardt, P. J. (1984). Bubble nucleation and the Coleman-Weinberg model. *Nucl. Phys. B* **237**, 176–188.

Linde, A. D. (1990). *Elementary particle physics and inflationary cosmology* [in russian], NAUKA, Moscow.

Linde, A. D. and Mezhlumian, A. (1995). Inflation with $\Omega \neq 1$. *Phys. Rev. D* **52**, 6789–6804.

Kofman, L., Linde, A. D., and Starobinsky, A. (1997). Towards the theory of reheating after inflation. *Phys. Rev. D* **56**, 3258–3295.

Kofman, L., Linde, A., and Starobinsky, A. (1994). Reheating after inflation. *Phys. Rev. Lett.* **73**, 3195–3198.

Samuel, D. A. and Hiscock, W. A. (1991). Effect of gravity on false-vacuum decay rates for O(4)-symmetric bubble nucleation. *Phys. Rev. D* **44**, 3052–3061.

Tanaka, T. (1999). The no-negative mode theorem in false vacuum decay with gravity. *Nucl. Phys. B* **556**, 373–396.